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The N-Widths of Hardy–Sobolev Spaces of Several Complex Variables

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Let B denote the unit ball in $\mathbb{C}^n$ with boundary S and let $\sigma(v)$ be the standard normalized measure on $S(B)$. For fixed $1 \leqslant p \leqslant \infty$, $R \geqslant 1$ let $BH^p(B_R)$ $(BA^p(B_R))$ denote the unit ball of the Hardy space H^p (resp. the Bergman space A^p) in $B_R := RB$ and for $l \in \mathbb{N}$ let $H_R(l, p, n)$ (resp. $A_R(l, p, n)$) denote the class of those functions which have the lth radial derivative belonging to $BH^p(B_R)$ $(BA^p(B_R))$; for $l=0$, let $H_R(0, p, n) := BH^p(B_R)$ $(A_R(0, p, n) := BA^p(B_R))$. The values of Kolmogorov, Gel'fand, and Bernstein and linear N-widths of classes $H_R(l, p, n)$ and $A_R(l, p, n)$ in the metrics $L^p(\sigma)$ and $L^p(v)$ (except for $A_R(l, p, n)$ in $L^p(\sigma)$) are found. For all $1 \leqslant p$, $q \leqslant \infty$, $R > 1$ the asymptotic estimates of N-widths for classes $H_R(l, p, n)$ and $A_R(l, p, n)$ in the spaces $L^q(\sigma)$ and $L^q(v)$ are also obtained.

1. INTRODUCTION AND STATEMENT OF RESULTS

Let X be a normed linear space and A be a convex, closed, centrally symmetric subset of X. The Kolmogorov d_N, Gel'fand d^N and linear δ_N N-widths of a set A in X are defined by

$$d_N(A; X) := \inf_{X_N} \sup_{x \in A} \inf_{y \in X_N} \|x - y\|, \qquad d^N(A; X) := \inf_{X^N} \sup_{x \in A \cap X^N} \|x\|,$$

$$\delta_N(A; X) := \inf_{A_N} \sup_{x \in A} \|x - A_N x\|,$$

where X_N (resp. X^N) runs over all N-dimensional (resp. N-codimensional) subspaces of X and A_N varies over all bounded linear operators of rank N which map X into itself. The Bernstein N-width of A in X is defined by

$$b_N(A; X) := \sup_{X_{N+1}} \sup\{r : rB(X_{N+1}) \subset A\},$$

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where $B(X_{N+1})$ is the unit ball of X_{N+1}. The standard reference for d_N, d^N, δ_N, and b_N is Pinkus [13]. For some additional information look at the review of Tikhomirov [18]. The detailed bibliography concerning N-widths of various classes of functions of one complex variable is given in [13, 16, 18]; see [5, 11, 19] for the case of several complex variables.

Let $B^n := \{z \in \mathbb{C}^n : |z| = (\sum_{j=1}^n |z_j|^2)^{1/2} < 1\}$, $S^n := \partial B^n$, $B_R^n := RB^n$, $S_R^n := \partial B_R^n$ $(R > 1)$, $U := B^1$, $T := \partial U$, $U_R := RU$, $T_R := \partial U_R$, and let v be the normalized Lebesgue measure in $\mathbb{C}^n = \mathbb{R}^{2n}$, $v(B^n) = 1$, σ be the probability measure on the sphere S^n which is invariant with respect to orthogonal group $O(2n)$ (see [14]). The Hardy spaces $H^p(B_R^n)$ (resp. the Bergman spaces $A^p(B_R^n)$) consist of all functions f holomorphic in B_R^n which have finite norms

$$\|f\|_{H^p(B_R^n)} := \sup_{0<r<R} \left(\int_{S^n} |f(r\zeta)|^p \, d\sigma(\zeta) \right)^{1/p}$$

$$\left(\text{resp.} \quad \|f\|_{A^p(B_R^n)} := \left(\int_{B_R^n} |f(z)|^p \, dv(z) \right)^{1/p} \right)$$

if $p < \infty$ and $\|f\|_{H^\infty(B_R^n)} = \|f\|_{A^\infty(B_R^n)} := \sup\{|f(z)| : z \in B_R^n\}$ if $p = \infty$. Let us denote by $BH^p(B_R^n)$ (resp. $BA^p(B_R^n)$) the closed unit ball in $H^p(B_R^n)$ (resp. $A^p(B_R^n)$).

If f is holomorphic in B_R^n with homogeneous polynomial expansion

$$f(z) = \sum_{m=0}^{\infty} F_m(z), \qquad z \in B_R^n, \tag{1.1}$$

then the radial derivative of f is defined by

$$\mathcal{R}f(z) := \sum_{m=1}^{\infty} mF_m(z)$$

(see [14]). For $l \in \mathbb{N}$ let

$$\mathcal{R}^l f(z) := \sum_{m=l}^{\infty} \frac{m!}{(m-l)!} F_m(z) \tag{1.2}$$

be the "lth radial derivative" of f (cf. [2]). For fixed $l, n \in \mathbb{N}$, $1 \leqslant p \leqslant \infty$, $R \geqslant 1$ the "Hardy–Sobolev space" $H_R(l, p, n)$ (resp. the space $A_R(l, p, n)$) consist of all functions f holomorphic in B_R^n for which $\mathcal{R}^l f \in BH^p(B_R^n)$ (resp. $\mathcal{R}^l f \in BA^p(B_R^n)$). Some results concerning these spaces are given in [2, 8, 16].

Let all $k=(k_1,...,k_n)\in\mathbb{Z}^n_+$ be numerated in such a way that $k=k(j)$, $|k(j)|\leqslant|k(j+1)|$ $(j=0,1,2,...)$, where as usual $|k|:=k_1+\cdots+k_n$. For $N\in\mathbb{N}$ let $N':=\min\{m\in\mathbb{N}\colon |k(m)|=|k(N)|\}$, $\tilde{N}:=|k(N)|$, and

$$\mathscr{P}_N(\mathbb{C}^n):=\operatorname{span}\{z^k\colon |k|\leqslant N, k\in\mathbb{Z}^n_+\},$$

$$\Pi_N(\mathbb{C}^n):=\operatorname{span}\{z^{k(j)}\colon j=0,1,...,N\},$$

$$\pi_N(\mathbb{C}^n):=\operatorname{span}\{z^{k(j)}\colon j=0,1,...,N'-1\},$$

where $z^k:=z_1^{k_1}\cdots z_n^{k_n}$. According to our notations $\pi_N(\mathbb{C}^n)=\Pi_{N'-1}(\mathbb{C}^n)$, $N'\leqslant N$, $|k(N'-1)|=\tilde{N}-1$, and

$$\tilde{N}=m \qquad \text{iff} \quad \binom{n+m-1}{n}\leqslant N\leqslant\binom{n+m}{n}-1$$

for m, $N\in\mathbb{N}$.

Now let f be given by (1.1) and let $l, N\in\mathbb{N}$, $l<\tilde{N}$, $\alpha^l_N:=(N-l)!/N!$. Then we set

$$(G_N f)(z):=\sum_{m=0}^{\tilde{N}-1}\left(1-\left(\frac{|z|}{R}\right)^{2(\tilde{N}-m)}\right)F_m(z), \tag{1.3}$$

$$(G^l_N f)(z):=\sum_{m=0}^{l-1}F_m(z)+\sum_{m=l}^{\tilde{N}-1}\left(1-\frac{\alpha^l_{2\tilde{N}-m}}{\alpha^l_m}\left(\frac{|z|}{R}\right)^{2(\tilde{N}-m)}\right)F_m(z), \tag{1.4}$$

and denote by $(P_N f)(z)$ and $(P^l_N f)(z)$ the right parts of (1.3) and (1.4) after replacing there $|z|$ by 1. Let

$$\mathscr{G}_N(\mathbb{C}^n):=\operatorname{span}\left\{z^{k(j)}\left(1-\left(\frac{|z|}{R}\right)^{2(\tilde{N}-|k(j)|)}\right)\colon j=0,1,...,N'-1\right\},$$

$$\mathscr{G}^l_N(\mathbb{C}^n):=\operatorname{span}\left\{\{z^{k(j)}\}_{j=0}^{l'},\left\{z^{k(j)}\left(1-\frac{\alpha^l_{2\tilde{N}-j}}{\alpha^l_j}\left(\frac{|z|}{R}\right)^{2(\tilde{N}-|k(j)|)}\right\}_{j=l'}^{N'-1}\right\},$$

where l' is defined as N'. In the case $l=0$ we set $\mathscr{G}^0_N(\mathbb{C}^n):=\mathscr{G}_N(\mathbb{C}^n)$, $G^0_N f:=G_N f$, $P^0_N f:=P_N f$, and $\mathscr{R}^0 f:=f$; then $H_R(0,p,n)=BH^p(B^n_R)$, $A_R(0,p,n)=BA^p(B^n_R)$.

The method $f\approx G^l_N f$ is optimal in the recovery problem of the value of the function $f\in H_R(l,\infty,1)$ at a given point $z\in U_R\setminus\{0\}$ by the Taylor information $\{f(0), f'(0), ..., f^{(N-1)}(0)\}$. This fact was noted by Osipenko [10] for $l=0$ and Donald J. Newman for $l=1$ (see Micchelli and Rivlin [9, p. 42]); it is not difficult to verify that the same property is true for all l.

Fisher and Micchelli [6] used the method of obtaining the upper bound for $\delta_N(BH^\infty(U_R), L^q(\mu))$, $R>1$, $1\leqslant q<\infty$, which coincide with $f\approx G_N f$ when $\mu=\nu$.

For $n=1$, $l \in \mathbb{N}$ the method $f \approx P_N^l f$ was found by Babenko [1] while solving the problem of determining the best approximation of functions $f \in H_R(l, \infty, 1)$ by polynomials of degree at most N. This method was applied by Taikov [17] and Pinkus [13, Chap. XIII] as well.

For $1 \leqslant p \leqslant \infty$, $N \in \mathbb{N}$ let

$$X_p^N(B^n) := \{f : f \in H^p(B^n), \partial f^{|k|}/\partial z^k = 0, k \in \{k(0), k(1), ..., k(N'-1)\}\},$$
$$Y_p^N(B^n) := \{f : f \in A^p(B^n), \partial f^{|k|}/\partial z^k = 0, k \in \{k(0), k(1), ..., k(N'-1)\}\},$$

The main result of this paper is the following:

THEOREM. *Let $1 \leqslant p \leqslant \infty$, $R \geqslant 1$, $l \in \mathbb{Z}_+$, $n, N \in \mathbb{N}$, $l < \tilde{N}$. Then*

$$d_N(H_R(l, p, n); L^p(\sigma)) = \alpha_{\tilde{N}}^l R^{\tilde{N}}, \tag{1.5}$$

$$d_N(H_R(l, p, n); L^p(v)) = \alpha_{\tilde{N}}^l R^{\tilde{N}} \left(\frac{p\tilde{N}}{2n} + 1\right)^{1/p}, \tag{1.6}$$

$$d_N(A_R(l, p, n); L^p(v)) = \alpha_{\tilde{N}}^l R^{\tilde{N} - 2n/p}, \tag{1.7}$$

and the same equalities are true for d^N, δ_N, b_N. Furthermore,

(a) *$\pi_N(\mathbb{C}^n)$ is an optimal subspace for $d_N(H_R(l, p, n); L^p(\sigma))$, $d_N(A_R(l, p, n); L^p(v))$, while $\mathcal{G}_N^l(\mathbb{C}^n)$ is optimal for $d_N(H_R(l, p, n); L^p(v))$.*

(b) *$X_p^N(B^n)$ is an optimal subspace for $d^N(H_R(l, p, n); L^p(\sigma))$, while $Y_p^N(B^n)$ is optimal for $d^N(H_R(l, p, n); L^p(v))$ and $d^N(A_R(l, p, n); L^p(v))$.*

(c) *P_N^l is an optimal operator for $\delta_N(H_R(l, p, n); L^p(\sigma))$ and $\delta_N(A_R(l, p, n); L^p(v))$, while G_N^l is optimal for $\delta_N(H_R(l, p, n); L^p(v))$.*

(d) *$\Pi_N(\mathbb{C}^n)$ is an optimal subspace for $b_N(H_R(l, p, n); L^p(\sigma))$, $b_N(H_R(l, p, n); L^p(v))$ and $b_N(A_R(l, p, n); L^p(v))$.*

When $n=1$ the statements of this theorem follow from the results of Babenko, Tikhomirov, Taikov, and Pinkus (see [13, p. 275]), except those connected with the equality (1.7) which was proved in [13] for $l=0$ only. It should be noted as classes $H_R(l, p, n)$ and $A_R(l, p, n)$ are defined by means of the radial (but not usual) derivative, the equalities (1.5)–(1.7) in the case $n=1$ differ from the corresponding ones in [13]. The theorem for $n>1$, $l=0$ was announced in [5].

The N-width d_N lies between b_N and δ_N:

$$b_N(A; X) \leqslant d_N(A; X) \leqslant \delta_N(A; X)$$

and d^N possesses the same property (see, e.g., [13, p. 207]). So the theorem will be proved if we obtain upper bounds for δ_N and related lower bounds for b_N. These estimates are established in Sections 2 and 3. We use the following two well known identities:

$$\int_{S^n} f(\zeta)\, d\sigma(\zeta) = \int_{S^n} d\sigma(\zeta) \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}\zeta)\, d\theta, \tag{1.8}$$

$$\int_{B_R^n} f(z)\, dv(z) = 2n \int_0^R r^{2n-1}\, dr \int_{S^n} f(r\zeta)\, d\sigma(\zeta). \tag{1.9}$$

In Section 4 we obtain the asymptotic estimates of the N-widths (d_N, d^N, δ_N, and b_N) for classes $H_R(l, p, n)$ and $A_R(l, p, n)$ in the metrics $L^q(\sigma)$ and $L^q(v)$ for all $1 \leqslant p, q \leqslant \infty$, $R > 1$.

Finally, in Section 5 we compare our estimates of $d_N(BH^\infty(B_R^n), L^q(v))$ with those recently obtained by Zakharyuta [19].

2. Upper Bounds for δ_N

For $0 < \rho < 1$, $t \in \mathbb{R}$ let

$$K_{l,N}(\rho, t) := \alpha_{\tilde N}^l + 2 \sum_{m=\tilde N+1}^{\infty} \rho^{m-\tilde N} \alpha_m^l \cos(\tilde N - m)\, t,$$

where $l \in \mathbb{Z}_+$, $N \in \mathbb{N}$, $l < \tilde N$. It is known, that

$$K_{l,N}(\rho, t) \geqslant 0 \tag{2.1}$$

for all $0 < \rho \leqslant 1$, $t \in \mathbb{R}$ (see [1; 13, p. 251]).

Let f be holomorphic in B_R^n and $0 < \rho < 1$, $\zeta \in S^n$; as in [14], f_ρ and $f_{\rho\zeta}$ are defined by $f_\rho(z) := f(\rho z)$, $z \in B_{R/\rho}^n$ and $f_{\rho\zeta}(\lambda) := f_\rho(\lambda\zeta)$, $\lambda \in U_{R/\rho}$. Then

$$f_\rho(z) - (G_N^l f_\rho)(z) = \frac{\lambda^l}{2\pi} \int_{-\pi}^{\pi} \left(\frac{r}{R}\right)^{\tilde N - l} \exp(i(l - \tilde N)(\Theta - \varphi))$$
$$\times K_{l,N}(r/R, \Theta - \varphi)\, f_{\rho\zeta}^{(l)}(Re^{i\Theta})\, d\Theta, \tag{2.2}$$

where $z = \lambda\zeta$, $\lambda = re^{i\varphi}$, $0 < r \leqslant R$, $\zeta \in S^n$ (cf. [13, p. 254]).

It is a known fact (e.g., [14]) that for any function $f \in H^p(B_R^n)$ there is a function f^* such that $f^*(R\zeta) = \lim_{r \to R-} f(r\zeta)$ a.e. in S^n and $\|f\|_{H^p(B_R^n)} = \|f^*(R\,\cdot)\|_{L^p(\sigma)}$. Further if $f \in H^p(B_R^n)$ then $f(R\zeta) := f^*(R\zeta)$ for a.e. $\zeta \in S^n$.

Proposition 2.1. *Let $1 \leqslant p \leqslant \infty$, $R \geqslant 1$, $l \in \mathbb{Z}_+$, $n, N \in \mathbb{N}$, $l < \tilde N$.*

(a) *If $f \in H_R(l, p, n)$ then*

$$\|f - P_N^l f\|_{L^p(\sigma)} \leqslant \alpha_{\tilde N}^l R^{-\tilde N} \tag{2.3}$$

and

$$\| f - G^l_N f \|_{L^p(\nu)} \leqslant \alpha^l_{\tilde{N}} R^{-\tilde{N}} \left(\frac{p\tilde{N}}{2n} + 1 \right)^{-1/p}. \tag{2.4}$$

(b) *If* $f \in A_R(l, p, n)$ *then*

$$\| f - P^l_N f \|_{L^p(\nu)} \leqslant \alpha^l_{\tilde{N}} R^{-\tilde{N}-(2n/p)}. \tag{2.5}$$

Proof. Suppose that f is holomorphic in B^n_R and let $1 \leqslant p < \infty$, $0 < \rho < 1$. Note that

$$\mathscr{R}^l f_\rho(\lambda\zeta) = \lambda^l f^{(l)}_{\rho\zeta}(\lambda)$$

for all $\lambda \in U_{R/\rho}$, $\zeta \in S^n$. It follows from (2.1) and (2.2) that

$$\int_{S^n} |f_\rho(\lambda\zeta) - (G^l_N f_\rho)(\lambda\zeta)|^p \, d\sigma(\zeta)$$
$$\leqslant \left(\frac{r}{R}\right)^{\tilde{N}p} \int_{S^n} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} K_{l,N}(r/R, \Theta - \varphi) \, |\mathscr{R}^l f_\rho(Re^{i\Theta}\zeta)| \, d\Theta \right)^p d\sigma(\zeta),$$

where $\lambda = re^{i\varphi}$, $0 < r \leqslant R$. Thus, since $\|K(r/R, \cdot)\|_{L^1(-\pi,\pi)} = \alpha^l_{\tilde{N}}$, identity (1.8) with the well known property of the convolution

$$\|h * g\|_{L^p(-\pi,\pi)} \leqslant \|h\|_{L^p(-\pi,\pi)} \cdot \|g\|_{L^1(-\pi,\pi)} \qquad (h \in L^p, g \in L^1)$$

gives

$$\| f_\rho(\lambda \cdot) - (G^l_N f_\rho)(\lambda \cdot) \|_{L^p(\sigma)} \leqslant \alpha^l_{\tilde{N}} \left(\frac{|\lambda|}{R} \right)^{\tilde{N}} \| \mathscr{R}^l f_\rho(R \cdot) \|_{L^p(\sigma)}, \tag{2.6}$$

where $\lambda \in \bar{U}_R$.

Let $f \in H_R(l, p, n)$. It follows from [8] that there is a q, $p \leqslant q \leqslant \infty$, such that $H_R(l, p, n) \subset H^q(B^n_R)$. Hence for our function f

$$\lim_{\rho \to 1-} \| f(\lambda \cdot) - f_\rho(\lambda \cdot) \|_{L^p(\sigma)} = 0$$

for all $\lambda \in T_R$ (see [14, Sect. 5.6.6]). In particular, the case $\lambda = R = 1$ is possible. If $\lambda \in U_R$, $\zeta \in S^n$, then by continuity $\lim_{\rho \to 1} f_\rho(\lambda\zeta) = f(\lambda\zeta)$. Also, it is easy to see that

$$\lim_{\rho \to 1-} G^l_N f_\rho = G^l_N f.$$

But then

$$\lim_{\rho \to 1-} \|f_\rho(\lambda\cdot) - (G^l_N f_\rho)(\lambda\cdot)\|_{L^p(\sigma)} = \|f(\lambda\cdot) - (G^l_N f)(\lambda\cdot)\|_{L^p(\sigma)}$$

and, since $\|\mathscr{R}^l f_\rho(R\cdot)\|_{L^p(\sigma)} \leqslant 1$, inequality (2.6) implies

$$\|f(\lambda\cdot) - (G^l_N f)(\lambda\cdot)\|_{L^p(\sigma)} \leqslant \alpha^l_{\tilde{N}} \left(\frac{|\lambda|}{R}\right)^{\tilde{N}}. \tag{2.7}$$

This immediately gives (2.3) by setting $\lambda = 1$ and substituting $P^l_N f$ for $G^l_N f$. If $\lambda = r$, $0 < r < R$, then the identity (1.9) with the inequality (2.7) gives (2.4).

Thus part (a) is established for $1 \leqslant p < \infty$.

Now let $f \in A_R(l, p, n)$. It follows from (2.6) that

$$\int_{S^n} |f(\rho\zeta) - (G^l_N f_\rho)(\zeta)|^p \, d\sigma(\zeta) \leqslant (\alpha^l_{\tilde{N}} R^{-\tilde{N}})^p \int_{S^n} |\mathscr{R}^l f(R\rho\zeta)|^p \, d\sigma(\zeta).$$

But by definition $(G^l_N f_\rho)(\cdot) = (P^l_N f_\rho)(\cdot)$ on S^n and identity (1.9) implies

$$\begin{aligned} \|f - P^l_N f\|^p_{L^p(v)} &\leqslant 2n(\alpha^l_{\tilde{N}} R^{-\tilde{N}})^p \int_0^1 \rho^{2n-1} \, d\rho \int_{S^n} |\mathscr{R}^l f(R\rho\zeta)|^p \, d\sigma(\zeta) \\ &= 2n(\alpha^l_{\tilde{N}})^p R^{-\tilde{N}p - 2n} \int_0^R r^{2n-1} \, dr \int_{S^n} |\mathscr{R}^l f(r\zeta)|^p \, d\sigma(\zeta) \\ &= (\alpha^l_{\tilde{N}})^p R^{-\tilde{N}p - 2n} \|\mathscr{R}^l f\|^p_{A^p(B^n_R)}, \end{aligned}$$

i.e., part (b) is proved for $1 \leqslant p < \infty$. In the case $p = \infty$, inequalities (2.3)–(2.5) coincide and obviously follow from (2.2).

The proof is complete.

COROLLARY 2.2. *If* $1 \leqslant p \leqslant \infty$, $R \geqslant 1$, $l \in \mathbb{Z}_+$, n, $N \in \mathbb{N}$, $l < \tilde{N}$, *then*

$$\delta_N(H_R(l, p, n); L^p(\sigma)) \leqslant \alpha^l_{\tilde{N}} R^{-\tilde{N}}, \tag{2.8}$$

$$\delta_N(H_R(l, p, n); L^p(v)) \leqslant \alpha^l_{\tilde{N}} R^{-\tilde{N}} \left(\frac{p\tilde{N}}{2n} + 1\right)^{-1/p}, \tag{2.9}$$

$$\delta_N(A_R(l, p, n); L^p(v)) \leqslant \alpha^l_{\tilde{N}} R^{-\tilde{N} - 2n/p} \tag{2.10}$$

This is a direct consequence of the preceding proposition and the definition of δ_N.

3. PROOF OF THE THEOREM

In this section the lower bounds for b_N inverse to (2.8)–(2.10) are derived and this completes the proof of the above formulated theorem.

PROPOSITION 3.1. *Let* $0<p\leqslant\infty$, $R>1$, n, $N\in\mathbb{N}$. *If* $P_N\in\mathscr{P}_N(\mathbb{C}^n)$, *then*

$$\|P_N(R\cdot)\|_{L^p(\sigma)}\leqslant R^N\,\|P_N\|_{L^p(\sigma)},\tag{3.1}$$

$$\|P_N\|_{L^p(\sigma)}\leqslant\left(\frac{Np}{2n}+1\right)^{1/p}\|P_N\|_{L^p(\nu)},\tag{3.2}$$

$$\|P_N(R\cdot)\|_{L^p(\nu)}\leqslant R^{N+2n/p}\,\|P_N\|_{L^p(\nu)}.\tag{3.3}$$

Proof. For $n=1$ inequality (3.1) is well known (see, e.g., [13, p. 252]). Let $P_N\in\mathscr{P}_N(\mathbb{C}^n)$, $0<p<\infty$, $n>1$. It is evident that $P_N(\lambda\zeta)$ for every fixed ζ in S^n is polynomial of degree N of the variable λ in $\mathbb{C}$. Thus, the one-dimensional variant of inequality (3.1) gives

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|P_N(Re^{i\Theta}\zeta)|^p\,d\Theta\leqslant\frac{R^{Np}}{2\pi}\int_{-\pi}^{\pi}|P_N(e^{i\Theta}\zeta)|^p\,d\Theta$$

for all $\zeta\in S$. Now, by the identity (1.8),

$$\begin{aligned}\int_S|P_N(R\zeta)|^p\,d\sigma(\zeta)&=\int_S d\sigma(\zeta)\frac{1}{2\pi}\int_{-\pi}^{\pi}|P_N(Re^{i\Theta}\zeta)|^p\,d\Theta\\&\leqslant R^{Np}\int_S d\sigma(\zeta)\frac{1}{2\pi}\int_{-\pi}^{\pi}|P_N(e^{i\Theta}\zeta)|^p\,d\Theta\\&=R^{Np}\int_S|P_N(\zeta)|^p\,d\sigma(\zeta).\end{aligned}$$

That is, (3.1) holds for each p, $0<p<\infty$. Also it is easy to see that for every $0<r<1$

$$r^{Np}\int_S|P_N(\zeta)|^p\,d\sigma(\zeta)\leqslant\int_S|P_N(r\zeta)|^p\,d\sigma(\zeta),$$

$$\int_S|P_N(Rr\zeta)|^p\,d\sigma(\zeta)\leqslant R^{Np}\int_S|P_N(r\zeta)|^p\,d\sigma(\zeta)$$

and the identity (1.9) gives (3.2), (3.3). The case $p=\infty$ is established by passing to the limit as $p\uparrow\infty$.

The proof of proposition is finished.

Remark. For $p=\infty$, inequality (3.1) represents the known Bernstein–Walsh inequality for a ball (see [15, p. 102]).

If $Q_N \in \mathscr{P}_N(\mathbb{C}^1)$, $R>0$, $1 \leqslant p < \infty$, then by the well-known Bernstein inequality

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |Q_N'(Re^{i\theta})|^p \, d\Theta \leqslant \left(\frac{N}{R}\right)^p \frac{1}{2\pi} \int_{-\pi}^{\pi} |Q_N(Re^{i\theta})|^p \, d\Theta \tag{3.4}$$

and $\|Q_N'\|_{C(T_R)} \leqslant (N/R) \, \|Q_N\|_{C(T_R)}$. (For a simple proof, see for instance [13, p. 252]).

Proposition 3.2. *Let $P_N \in \mathscr{P}_N(\mathbb{C}^n)$, $R>0$, $1 \leqslant p \leqslant \infty$, l, n, $N \in \mathbb{N}$, $l<N$. Then*

$$\|\mathscr{R}^l P_N\|_{H^p(B_R^n)} \leqslant (\alpha_N^l)^{-1} \, \|P_N\|_{H^p(B_R^n)} \tag{3.5}$$

and

$$\|\mathscr{R}^l P_N\|_{A^p(B_R^n)} \leqslant (\alpha_N^l)^{-1} \, \|P_N\|_{A^p(B_R^n)}. \tag{3.6}$$

Proof. It is sufficient to consider the case $1 \leqslant p < \infty$. For $\lambda \in \mathbb{C}$, $\zeta \in S^n$ let $P_{N\zeta}(\lambda) := P_N(\lambda\zeta)$. From the definition of the lth radial derivative (see (1.2)) it follows that

$$\mathscr{R}^l P_N(\lambda\zeta) = \lambda^l P_{N\zeta}^{(l)}(\lambda).$$

Hence, applying (3.4) to $P_{N\zeta}$ yields

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |\mathscr{R}^l P_N(Re^{i\theta}\zeta)|^p \, d\Theta \leqslant \frac{(\alpha_N^l)^{-p}}{2\pi} \int_{-\pi}^{\pi} |P_N(Re^{i\theta}\zeta)|^p \, d\Theta.$$

Combining this with the identity (1.8), we obtain

$$\|\mathscr{R}^l P_N(R\cdot)\|_{L^p(\sigma)} \leqslant (\alpha_N^l)^{-1} \, \|P_N(R\cdot)\|_{L^p(\sigma)}.$$

That is, (3.5) holds. Now by (1.9) and (3.5) we get

$$\begin{aligned}\int_{B_R^n} |\mathscr{R}^l P_N(z)|^p \, dv(z) &= 2n \int_0^R r^{2n-1} \, dr \int_{S^n} |\mathscr{R}^l P_N(r\zeta)|^p \, d\sigma(\zeta) \\ &\leqslant 2n(\alpha_n^l)^{-p} \int_0^R r^{2n-1} \, dr \int_{S^n} |P_N(r\zeta)|^p \, d\sigma(\zeta) \\ &= (\alpha_N^l)^{-p} \int_{B_R^n} |P_N(z)|^p \, dv(z),\end{aligned}$$

which establishes the proposition.

COROLLARY 3.3. *If* $1 \leqslant p \leqslant \infty$, $R \geqslant 1$, $l \in \mathbb{Z}_+$, n, $N \in \mathbb{N}$, $l < \tilde{N}$, *then*

$$b_N(H_R(l, p, n); L^p(\sigma)) \geqslant \alpha^l_{\tilde{N}} R^{-\tilde{N}}, \tag{3.7}$$

$$b_N(H_R(l, p, n); L^p(v)) \geqslant \alpha^l_{\tilde{N}} R^{-\tilde{N}} \left(\frac{p\tilde{N}}{2n} + 1\right)^{-1/p}, \tag{3.8}$$

$$b_N(A_R(l, p, n); L^p(v)) \geqslant \alpha^l_{\tilde{N}} R^{-\tilde{N} - 2n/p}. \tag{3.9}$$

Indeed, according to (3.1) and (3.5), if

$$P_N \in \Pi_N(\mathbb{C}^n), \qquad \|P_N\|_{L^p(\sigma)} \leqslant \alpha^l_{\tilde{N}} R^{-\tilde{N}},$$

then $P_N \in H_R(l, p, n)$. That is, (3.7) holds. Also, it is easy to see that Propositions 3.1 and 3.2 imply the inequalities (3.8) and (3.9).

Proof of the Theorem. As noted in Section 1, $b_N \leqslant d_N \leqslant \delta_N$ and $b_N \leqslant d^N \leqslant \delta_N$ are always true. That is why the required equalities for N-widths are derived from Corollaries 2.2 and 3.3. Now parts (a), (c), (d) of the theorem come out from the methods of proofs of these corollaries. Part (b) for $d^N(H_R(l, p, n); L^p(\sigma))$ follows from the fact that

$$\sup\{\|f\|_{L^p(\sigma)} : f \in X^N_p(B^n)\} = \sup\{\|f - P^l_N f\|_{L^p(\sigma)} : f \in X^N_p(B^n)\} \leqslant \alpha^l_{\tilde{N}} R^{-\tilde{N}}.$$

The optimality of the subspaces $Y^N_p(B^n)$ for $d^N(H_R(l, p, n); L^p(v))$ and $d^N(A_R(l, p, n); L^p(v))$ is likewise established.

This completes the proof of the theorem.

4. SOME ASYMPTOTIC ESTIMATES

For $n = 1$ the asymptotic estimates for the N-widths (d_N, d^N, δ_N and b_N) of the classes $H_R(l, p, n)$ and $A_R(l, p, n)$ in the metrics $L^q(\sigma)$ and $L^q(v)$ for all $1 \leqslant p$, $q \leqslant \infty$, $R > 1$ come out from the result of [4]. These estimates will now be extended to the multidimensional case.

We recall that, if $x_N > 0$ and $y_N > 0$ for all $N \in \mathbb{N}$ then the notation $x_N \asymp y_N$ means that there exist positive constants c_1 and c_2, independent of N, for which $c_1 y_N \leqslant x_N \leqslant c_2 y_N$ for all N sufficiently large.

PROPOSITION 4.1. *Let* $1 \leqslant p$, $q \leqslant \infty$, $R > 1$, $l \in \mathbb{Z}_+$, $n \in \mathbb{N}$. *Then*

$$d_N(A_R(l, p, n); L^q(v)) \asymp \tilde{N}^{-l + 1/p - 1/q} R^{-\tilde{N}}, \tag{4.1}$$

$$d_N(A_R(l, p, n); L^q(\sigma)) \asymp \tilde{N}^{-l + 1/p} R^{-\tilde{N}}, \tag{4.2}$$

$$d_N(H_R(l, p, n); L^q(v)) \asymp \tilde{N}^{-l - 1/q} R^{-\tilde{N}}, \tag{4.3}$$

$$d_N(H_R(l, p, n); L^q(\sigma)) \asymp \tilde{N}^{-l} R^{-\tilde{N}}, \tag{4.4}$$

and the same relations are true for d^N, δ_N, b_N.

Proof. If f is holomorphic in B_R^n with homogeneous expansion (1.1), then

$$F_m(\lambda\zeta) = \lambda^m c_m(f_\zeta), \qquad \lambda \in U_R, \quad \zeta \in S^n, \tag{4.5}$$

where

$$f_\zeta(\lambda) := f(\lambda\zeta) = \sum_{m=0}^{\infty} c_m(f_\zeta)\, \lambda^m.$$

For $m > l$, $0 < r < R$,

$$c_m(f_\zeta) = \frac{\alpha_m^l}{2\pi i} \int_{T_r} \frac{f_\zeta^{(l)}(\lambda)}{\lambda^{m-l+1}}\, d\lambda = \frac{\alpha_m^l}{2\pi i} \int_{T_r} \frac{\mathscr{R}^l f(\lambda\zeta)}{\lambda^{m+1}}\, d\lambda.$$

Let $f \in A_R(l, p, n)$, $1 \leqslant p < \infty$. Hölder's inequality gives

$$|c_m(f_\zeta)| \leqslant \alpha_m^l \left(\frac{1}{2\pi} \int_{T_r} |\mathscr{R}^l f(re^{i\Theta}\zeta)|^p \, d\Theta \right)^{1/p}$$

and (1.8) and (1.9) imply that

$$|c_m(f_\zeta)|\, r^m \leqslant \alpha_m^l \left(\frac{mp}{2n} + 1 \right)^{1/p} R^{-m-2n/p}\, \|\mathscr{R}^l f\|_{A^p(B_R^n)}. \tag{4.6}$$

Let $N \in \mathbb{N}$, $\tilde{N} > l$, $1 \leqslant q < \infty$. Since $\|\mathscr{R}^l f\|_{A^p(B_R^n)} \leqslant 1$, by (1.9), (4.5), and (4.6) it follows that

$$\begin{aligned} \left\| f - \sum_{m=0}^{\tilde{N}-1} F_m \right\|_{L^q(v)}^q &= 2n \int_0^1 r^{2n-1}\, dr \int_{S^n} \left| \sum_{m=\tilde{N}}^{\infty} F_m(r\zeta) \right|^q d\sigma(\zeta) \\ &\leqslant c_3 \int_0^1 \left(\sum_{m=\tilde{N}}^{\infty} m^{-l+1/p} R^{-m} r^m \right)^q r^{2n-1}\, dr \\ &\leqslant c_4 \tilde{N}^{-lq+q/p-1} R^{-\tilde{N}q}. \end{aligned}$$

Also, using (1.8), (4.5), and (4.6), it is easy to see that

$$\left\| f - \sum_{m=0}^{\tilde{N}-1} F_m \right\|_{L^q(\sigma)} \leqslant c_5 \tilde{N}^{-l+1/p} R^{-\tilde{N}}$$

for all $1 \leqslant q \leqslant \infty$. Hence,

$$\delta_N(A_R(l, p, n); L^q(v)) \leqslant c_6 \tilde{N}^{-l+1/p-1/q} R^{-\tilde{N}}, \tag{4.7}$$

$$\delta_N(A_R(l, p, n); L^q(\sigma)) \leqslant c_7 \tilde{N}^{-l+1/p} R^{-\tilde{N}}. \tag{4.8}$$

Let $Q \in \mathscr{P}_N(\mathbb{C}^n)$, $1 \leqslant p, q \leqslant \infty$. We show that for every $R > 1$ there is a constant c_8, independent of R and Q, such that

$$\|Q\|_{A^p(B^n_R)} \leqslant c_8 N^{-1/p} R^N \|Q\|_{L^q(\sigma)}. \tag{4.9}$$

If $Q = \sum_{m=0}^N Q_m$ is the homogeneous expansion of Q and

$$Q_\zeta(\lambda) := Q(\lambda\zeta) = \sum_{m=0}^{N} c_m(Q_\zeta)\, \lambda^m, \qquad \lambda \in \mathbb{C}, \quad \zeta \in S^n,$$

then by (4.5)

$$Q_m(\lambda\zeta) = \lambda^m c_m(Q_\zeta).$$

By the formula for the Taylor's coefficients

$$|c_m(Q_\zeta)| \leqslant \frac{1}{2\pi} \int_{-\pi}^{\pi} |Q(e^{i\Theta}\zeta)|\, d\Theta.$$

Now, by Hölder's inequality and the identity (1.8)

$$|c_m(Q_\zeta)| \leqslant \|Q\|_{L^q(\sigma)}$$

for all $1 \leqslant q \leqslant \infty$. But then

$$\begin{aligned}
\|Q\|^p_{A^p(B^n_R)} &= 2n \int_0^R r^{2n-1}\, dr \int_{S^n} \left| \sum_{m=0}^{N} Q_m(r\zeta) \right|^p d\sigma(\zeta) \\
&\leqslant c_9 \|Q\|^p_{L^q(\sigma)} \int_0^R \left(\sum_{m=0}^{N} r^m \right) r^{2n-1}\, dr \\
&\leqslant c_{10} N^{-1} R^{Np} \|Q\|^p_{L^p(\sigma)}.
\end{aligned}$$

That is, (4.9) holds.

Now let $P_N \in \Pi_{N+1}(\mathbb{C}^n)$. Then by (4.9) and Propositions 3.1 and 3.2,

$$\begin{aligned}
\|\mathscr{R}^l P_N\|_{A^p(B^n_R)} &\leqslant c_{11} R^{\tilde N} \tilde N^{-1/p} \|\mathscr{R}^l P_N\|_{L^q(\sigma)} \leqslant c_{12} R^{\tilde N} \tilde N^{l-1/p} \|P_N\|_{L^q(\sigma)} \\
&\leqslant c_{13} R^{\tilde N} \tilde N^{l-1/p+1/q} \|P_N\|_{L^q(v)}.
\end{aligned}$$

It follows that

$$\begin{aligned}
b_N(A_R(l, p, n); L^q(v)) &\geqslant c_{14} \tilde N^{-l+1/p-1/q} R^{-\tilde N}, \\
b_N(A_R(l, p, n); L^q(\sigma)) &\geqslant c_{15} \tilde N^{-l+1/p} R^{-\tilde N},
\end{aligned}$$

which jointly with (4.7) and (4.8) give (4.1) and (4.2). Relations (4.3) and (4.4) are proved by the same method.

The proof is finished.

Remark. If $1 \leqslant q \leqslant p \leqslant \infty$, $R \geqslant 1$, $N \in \mathbb{N}$, then

$$d_N(BH^p(U_R), L^q(\sigma)) = R^{-N}, \qquad d_N(BH^p(U_R), L^q(\nu)) = R^{-N}\left(\frac{qN}{2}+1\right)^{-1/q},$$

and both equalities are true for d^N and δ_N (see [5, 12]). For $p < q$ exact values are known only for the N-widths (d^N and δ_N) of $BH^2(U_R)$ in $C(T)$ (see [11]).

5. Concluding Notes

Let $BH^\infty(\Omega)$ denote the class of those functions f which are holomorphic in the domain $\Omega \subset \mathbb{C}^n$ and satisfy $|f| \leqslant 1$ therein. Let K be a compact subset of Ω. The history of the widths for the class $BH^\infty(\Omega)$ in $C(K)$ in the case $n = 1$ can be found in [18] (see also [13, p. 276]). For $n > 1$, Zakharyuta [19] has recently got the asymptotic formula

$$\log d_N(BH^\infty(\Omega), C(K)) \sim -2\pi\left(\frac{n!}{C(K, \Omega)}\right)^{1/n} N^{1/n} \qquad (n \to \infty), \tag{5.1}$$

where compact $K \subset \Omega$ is subjected to some conditions of regularity and $C(K, \Omega)$ is the capacity of K related to Ω. Here the notation $x_N \sim y_N$ $(N \to \infty)$ means that $\lim_{N \to \infty} (x_N/y_N) = 1$. The proof of formula (5.1) is obtained by extension of the methods of the paper [20] to the multidimensional case using complex potential theory.

In the case when $\Omega = G_R$ is a canonical neighbourhood of a compact K in $\mathbb{C}^n$, formula (5.1) assumes the form

$$\log d_N(BH^\infty(G_R), C(K)) \sim -(n!\,N)^{1/n} \log R \qquad (R > 1, N \to \infty) \tag{5.2}$$

and can be derived from the multivariate Bernstein–Walsh theorem (see, e.g., [15, Chap. 3]).

For $p = \infty$, $l = 0$, equality (1.5) gives

$$d_N(BH^\infty(B^n_R), C(\bar{B}^n)) = R^{-\tilde{N}}. \tag{5.3}$$

Formula (5.3) is in agreement with (5.1) and (5.2), as $\tilde{N} \sim (n!\,N)^{1/n}$ $(N \to \infty)$ and $C(\bar{B}^n, B^n_R) = (2\pi/\log R)^n$. By Proposition 4.1

$$d_N(BH^\infty(B^n_R), L^q(\nu)) \asymp \tilde{N}^{-1/q} R^{-\tilde{N}}. \tag{5.4}$$

for $1 \leqslant q < \infty$.

Problem. Comparing (5.4) with (5.1) and (5.3) it is natural to look for conditions on K and Ω, $K \subset \Omega \subset \mathbb{C}^n$, where

$$d_N(BH^\alpha(\Omega), L^q(K)) \asymp \tilde{N}^{-1/q} \exp(-\tilde{N}\alpha), \qquad (5.5)$$

with $1 \leqslant q \leqslant \infty$, $\alpha := 2\pi/(C(K, \Omega))^{1/n}$. If $n = 1$ and the boundary ∂K consists of a finite number of disjoint curves of bounded rotation, then formula (5.5) can be proved (at least in case $q = \infty$) by the methods considered in [3, 7]. In both methods upper bounds are received by some modifications of the classical Faber approximation. It is thus natural to ask which methods will be used in the multidimensional case.

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